

## ON THE CONTINUOUS BINOMIAL COEFFICIENTS OF CANO AND DIAZ

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ABSTRACT. Here are exhibited some additional results about the continuous binomial coefficients as introduced by L. Cano and R. Diaz in [1].

## 1. A GENERAL RESULT

In [1], Cano and Diaz introduce a continuous extension of the usual binomial coefficients as follows

$$\left\{ \begin{matrix} x \\ s \end{matrix} \right\} = 2I_0 \left( 2\sqrt{s(x-s)} \right) + \frac{x}{\sqrt{s(x-s)}} I_1 \left( 2\sqrt{s(x-s)} \right), \quad 0 \leq s \leq x$$

where  $I_\nu$  is the modified Bessel function.

In the same way as the usual binomial coefficient  $\binom{m}{n}$  counts the subsets of cardinality  $n$  of a set of cardinality  $m$ , the continuous binomial coefficient  $\left\{ \begin{matrix} x \\ s \end{matrix} \right\}$  is equal to the volume of the space of directed (continuous) paths  $\Gamma(s, x-s)$  that start at the origin with travel time  $s$  in the horizontal direction and  $x-s$  in the vertical direction (see [1] for more details).

We start with a general result that allows to compute several quantities associated with these coefficients.

**Theorem 1.** *The function*

$$I_\Phi(x) = \int_0^x \left\{ \begin{matrix} x \\ s \end{matrix} \right\} \Phi(s) ds$$

has Laplace transform

$$\int_0^{+\infty} I_\Phi(x) e^{-px} dx = \left( \frac{1+p}{p} \right)^2 \tilde{\Phi} \left( p - \frac{1}{p} \right) - \tilde{\Phi}(p)$$

where  $\tilde{\Phi}(p)$  is the Laplace transform of  $\Phi(s)$ .

As a consequence,

$$(1.1) \quad I_\Phi(x) = \mathcal{L}^{-1} \left[ \left( \frac{1+p}{p} \right)^2 \tilde{\Phi} \left( p - \frac{1}{p} \right) \right] - \Phi(x)$$

where  $\mathcal{L}^{-1}$  denotes the inverse Laplace transform.

*Proof.* We apply Fubini's theorem to transform the double integral

$$\begin{aligned} \int_0^{+\infty} I_\Phi(x) e^{-px} dx &= \int_0^{+\infty} \int_0^x \left\{ \begin{matrix} x \\ s \end{matrix} \right\} \Phi(s) ds e^{-px} dx \\ &= \int_0^{+\infty} \Phi(s) \int_s^{+\infty} e^{-px} \left\{ \begin{matrix} x \\ s \end{matrix} \right\} dx ds. \end{aligned}$$

The inner integral is now evaluated using the change of variable  $x = s + w$  as

$$\int_s^{+\infty} e^{-px} \left\{ \begin{matrix} x \\ s \end{matrix} \right\} dx = e^{-sp} \int_0^{+\infty} e^{-pw} \left\{ \begin{matrix} s+w \\ s \end{matrix} \right\} dw.$$

Using the expression of the binomial coefficient, we deduce

$$\begin{aligned} \int_0^{+\infty} e^{-pw} \left\{ \begin{matrix} s+w \\ s \end{matrix} \right\} dw &= \int_0^{+\infty} e^{-pw} \left[ 2I_0(2\sqrt{sw}) + \frac{w+s}{\sqrt{sw}} I_1(2\sqrt{sw}) \right] dw \\ &= 2 \int_0^{+\infty} e^{-pw} I_0(2\sqrt{sw}) dw + \int_0^{+\infty} e^{-pw} \frac{w+s}{\sqrt{sw}} I_1(2\sqrt{sw}) dw. \end{aligned}$$

These Laplace transforms can be found in [2, 6.614.3 and 6.643.2] and evaluate respectively as

$$\int_0^{+\infty} e^{-pw} I_0(2\sqrt{sw}) dw = \frac{1}{p} e^{\frac{s}{p}},$$

$$\int_0^{+\infty} e^{-pw} \frac{w}{\sqrt{sw}} I_1(2\sqrt{sw}) dw = \frac{1}{p^2} e^{\frac{s}{p}}$$

and

$$\int_0^{+\infty} e^{-pw} \frac{1}{\sqrt{sw}} I_1(2\sqrt{sw}) dw = -1 + e^{\frac{s}{p}}.$$

We deduce

$$\int_0^{+\infty} e^{-pw} \left\{ \begin{matrix} s+w \\ s \end{matrix} \right\} dw = e^{\frac{s}{p}} \frac{p^2 + 2p + 1}{p^2} - 1 = e^{\frac{s}{p}} \left( \frac{p+1}{p} \right)^2 - 1.$$

This is now substituted in the outer integral to obtain

$$\int_0^{+\infty} \Phi(s) e^{-sp} \left\{ e^{\frac{s}{p}} \left( \frac{p+1}{p} \right)^2 - 1 \right\} ds = \left( \frac{p+1}{p} \right)^2 \int_0^{+\infty} \Phi(s) e^{-s(p-\frac{1}{p})} ds - \int_0^{+\infty} \Phi(s) e^{-sp} ds.$$

These two integral are recognized as the Laplace transforms of  $\Phi(s)$  computed respectively at  $p - \frac{1}{p}$  and  $p$ , and the result follows.  $\square$

A first consequence of this result is as follows.

**Corollary 2.** *Choosing  $\Phi(s) = \alpha^s e^{us}$ , we deduce the value of the integral*

$$\begin{aligned} (1.2) \quad \int_0^x \left\{ \begin{matrix} x \\ s \end{matrix} \right\} \alpha^s e^{us} ds &= 2\alpha^{\frac{x}{2}} e^{\frac{ux}{2}} \left\{ \cosh \left( \frac{x}{2} \sqrt{4 + (u + \log \alpha)^2} \right) - \cosh \left( \frac{x}{2} (u + \log \alpha) \right) \right. \\ &\quad \left. + \frac{2}{\sqrt{4 + (u + \log \alpha)^2}} \sinh \left( \frac{x}{2} \sqrt{4 + (u + \log \alpha)^2} \right) \right\}. \end{aligned}$$

*Proof.* Choose  $\Phi(s) = \alpha^s e^{us}$  so that  $\tilde{\Phi}(p) = \frac{1}{p - u - \log \alpha}$  and use formula (1.1) to obtain the result.  $\square$

Another consequence is as follows.

**Corollary 3.** *The integral*

$$\int_0^x \left\{ \begin{matrix} x \\ s \end{matrix} \right\} ds = 2(e^x - 1)$$

*as computed in [1].*

*Proof.* We recover this value by choosing  $\Phi(s) = 1$  in the previous theorem. This gives  $\tilde{\Phi}(p) = \frac{1}{p}$  and the integral is deduced as the inverse Laplace transform

$$\mathcal{L}^{-1} \left[ \left( \frac{p+1}{p} \right)^2 \frac{1}{p - \frac{1}{p}} - \frac{1}{p} \right] = \mathcal{L}^{-1} \left[ \frac{2}{p-1} - \frac{1}{p} \right] - 1 = 2(e^x - 1).$$

$\square$

## 2. THE CONTINUOUS BINOMIAL DISTRIBUTION

The new continuous binomial coefficients allow to define a continuous version of the discrete binomial distribution as

$$(2.1) \quad f_{x,p}(s) = \frac{1}{A_{x,p}} \left\{ \begin{matrix} x \\ s \end{matrix} \right\} p^s (1-p)^{x-s}, \quad 0 \leq s \leq x$$

where  $0 \leq p \leq 1$  and the normalization constant  $A_{x,p}$  is such that

$$\int_0^x f_x(s) ds = 1.$$

This constant is not evaluated in [1]: we give its value as follows.

**Theorem 4.** *The normalization constant of the continuous binomial distribution is equal to*

$$A_{x,p} = 2 [p(1-p)]^{\frac{x}{2}} \left\{ \cosh \left( \frac{x}{2} \sqrt{4 + \log^2 \frac{p}{1-p}} \right) - \cosh \left( \frac{x}{2} \log \frac{p}{1-p} \right) \right. \\ \left. + \frac{2}{\sqrt{4 + \log^2 \frac{p}{1-p}}} \sinh \left( \frac{x}{2} \sqrt{4 + \log^2 \frac{p}{1-p}} \right) \right\}$$

*Proof.* Since

$$A_{x,p} = \int_0^x \left\{ \begin{matrix} x \\ s \end{matrix} \right\} p^s (1-p)^{x-s} ds = (1-p)^x \int_0^x \left\{ \begin{matrix} x \\ s \end{matrix} \right\} \left( \frac{p}{1-p} \right)^s ds,$$

use (1.2) with  $\alpha = \frac{p}{1-p}$  and  $u = 0$  to obtain the result.  $\square$

The moment generating function can also be computed explicitly as follows.

**Theorem 5.** *The moment generating function of the continuous binomial distribution (2.1) is*

$$(2.2) \quad \mathbb{E} e^{uX} = e^{\frac{ux}{2}} \frac{\varphi_{x,p}(u)}{\varphi_{x,p}(0)}$$

with

$$\varphi_{x,p}(u) = \cosh \left( \frac{x}{2} \sqrt{4 + \left( u + \log \frac{p}{1-p} \right)^2} \right) - \cosh \left( \frac{x}{2} \log \frac{p}{1-p} \right) \\ + \frac{2}{\sqrt{4 + \left( u + \log \frac{p}{1-p} \right)^2}} \sinh \left( \frac{x}{2} \sqrt{4 + \left( u + \log \frac{p}{1-p} \right)^2} \right).$$

The centered version of this distribution is defined in [1] as

$$(2.3) \quad f_{x,p}(s) = \frac{1}{A_{x,p}} \left\{ \begin{matrix} x \\ \frac{x}{2} + s \end{matrix} \right\} p^{s+\frac{x}{2}} (1-p)^{\frac{x}{2}-s}, \quad -\frac{x}{2} \leq s \leq \frac{x}{2};$$

remark that this is the distribution of the shifted random variable

$$Y = X - \frac{x}{2}$$

where  $X$  is distributed as in (2.1). All its moments can be explicitly computed in the case  $p = \frac{1}{2}$  as follows.

**Theorem 6.** *Let  $Y$  a random variable with centered binomial distribution as in (2.3) with  $p = \frac{1}{2}$ . Its moment generating function is*

$$(2.4) \quad \mathbb{E} e^{uY} = \frac{\cosh \left( \frac{x}{2} \sqrt{4 + u^2} \right) - \cosh \left( \frac{x}{2} u \right) + \frac{2}{\sqrt{4 + u^2}} \sinh \left( \frac{x}{2} \sqrt{4 + u^2} \right)}{e^x - 1}.$$

We deduce that the odd moments of  $Y$  are equal to zero while the even moments are given by

$$\mathbb{E}Y^{2k} = \frac{\sqrt{\pi} \frac{2k!}{k!2^{3k+\frac{1}{2}}} x^{k+\frac{1}{2}} \left( I_{k-\frac{1}{2}}(x) + I_{k+\frac{1}{2}}(x) \right) - \left( \frac{x}{2} \right)^{2k}}{e^x - 1}.$$

As a particular case, the variance is given by

$$\mathbb{E}Y^2 = \frac{1}{4} \frac{x(e^x - x) - \sinh x}{e^x - 1}.$$

*Proof.* Since  $Y = X - \frac{1}{2}$  and  $p = \frac{1}{2}$ , the moment generating function of  $Y$  is deduced from (2.2) as

$$\mathbb{E}e^{uY} = \frac{\varphi_{x, \frac{1}{2}}(u)}{\varphi_{x, \frac{1}{2}}(0)}$$

which gives (2.4). The moments are deduced from the Taylor expansion of this function at  $u = 0$ . In particular, since this is an even function, all the odd moments are equal to 0.  $\square$

### 3. CONTINUOUS CATALAN NUMBERS

The continuous Catalan numbers are defined in [1] as follows

$$(3.1) \quad C(x, y) = \sum_{n=0}^{+\infty} \text{vol}(\Lambda^n(x, y)), \quad 0 \leq y \leq x$$

where  $\Lambda^n(x, y)$  is the set of directed paths joining  $(0, 0)$  to  $(x, y)$  with pattern of length  $2n + 2$  (see [1, Section 5]).

These numbers can be computed explicitly as follows.

**Theorem 7.** *The continuous Catalan numbers defined by (3.1) are equal to*

$$(3.2) \quad C(x, y) = I_0\left(\sqrt{x^2 - y^2}\right) - \frac{x - y}{x + y} I_2\left(\sqrt{x^2 - y^2}\right).$$

The proof of this result requires the following lemma.

**Lemma 8.** *For  $n \geq 0$ , the volume  $\text{vol}(\Lambda^n(x, y))$  is equal to*

$$(3.3) \quad \text{vol}(\Lambda^n(x, y)) = \frac{(x - y)^n (x + y)^{n-1} (x + (2n + 1)y)}{2^{2n} n! (n + 1)!}.$$

*Proof.* The proof of this lemma is elementary by showing that the right hand side sequence in (3.3) satisfies the recursion given in [1, Prop. 27]

$$\text{vol}(\Lambda^{n+1}(x, y)) = \int_0^{\frac{x-y}{2}} \int_0^{\frac{x+y}{2}-b} \text{vol}(\Lambda^n(a + 2b, a)) da db$$

and the initial condition  $\text{vol}(\Lambda^0(x, y)) = 1$ .  $\square$

The proof of Theorem 7 is now deduced by computing the sum in (3.1) using the expression (3.3).

It is shown in [1, Prop. 29] that the Catalan numbers satisfy the integral equation given in

$$(3.4) \quad C(x, y) = 1 + \int_0^{\frac{x-y}{2}} \int_0^{\frac{x+y}{2}-b} C(a + 2b, a) da db.$$

As a check of our result, we verify the following claim.

**Proposition 9.** *The Catalan numbers  $C(x, y)$  defined by (3.2) satisfy the integral equation (3.4).*

*Proof.* From (3.2), we deduce after simplification

$$C(a + 2b, a) = \frac{I_1\left(2\sqrt{b(a+b)}\right)}{\sqrt{b(a+b)}} + \frac{a}{a+b} I_2\left(2\sqrt{b(a+b)}\right).$$

The double integral

$$\int_0^{\frac{x-y}{2}} \int_0^{\frac{x+y}{2}-b} \frac{I_1\left(2\sqrt{b(a+b)}\right)}{\sqrt{b(a+b)}} + \frac{a}{a+b} I_2\left(2\sqrt{b(a+b)}\right) da db$$

is easily computed as

$$-1 + I_0\left(\sqrt{x^2 - y^2}\right) - \frac{x-y}{x+y} I_2\left(\sqrt{x^2 - y^2}\right)$$

which, using the identity for contiguous Bessel functions

$$(3.5) \quad I_0\left(\sqrt{x^2 - y^2}\right) - I_2\left(\sqrt{x^2 - y^2}\right) = \frac{2}{\sqrt{x^2 - y^2}} I_1\left(\sqrt{x^2 - y^2}\right),$$

gives the final result.  $\square$

We remark that the use of identity (3.5) allows to express the continuous Catalan numbers equivalently as

$$\begin{aligned} C(x, y) &= I_0\left(\sqrt{x^2 - y^2}\right) - \frac{x-y}{x+y} \left( I_0\left(\sqrt{x^2 - y^2}\right) - \frac{2}{\sqrt{x^2 - y^2}} I_1\left(\sqrt{x^2 - y^2}\right) \right) \\ &= 2 \left( \frac{y}{x+y} I_0\left(\sqrt{x^2 - y^2}\right) + \frac{x-y}{x+y} \frac{I_1\left(\sqrt{x^2 - y^2}\right)}{\sqrt{x^2 - y^2}} \right). \end{aligned}$$

The special case  $y = 0$  gives the continuous Catalan function as defined in[1]:

$$C(2x, 0) = \frac{I_1(2x)}{x}.$$

Moreover,  $C_n$  denoting the usual Catalan numbers,

$$\frac{I_1(2x)}{x} = \sum_{n \geq 0} \frac{x^{2n}}{n!(n+1)!} = \sum_{n \geq 0} \frac{x^{2n}}{2n!} C_n,$$

so that the continuous Catalan function  $C(2x, 0)$  is related to the generating function of Catalan numbers as

$$\int_0^{+\infty} C(2\sqrt{x}u, 0) e^{-u} du = \sum_n C_n x^n = \frac{2}{1 + \sqrt{1 - 4x^2}}.$$

The fact that a discrete generating function of the Catalan numbers is related to a continuous integral transform of the continuous Catalan function should not come as a surprise.

## REFERENCES

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